Problem 4.23

(a) Prove that for a particle in a potential $V(\mathbf{r})$ the rate of change of the expectation value of the orbital angular momentum $\mathbf{L}$ is equal to the expectation value of the torque:

$$\frac{d}{dt}\langle \mathbf{L} \rangle = \langle \mathbf{N} \rangle,$$

where

$$\mathbf{N} = \mathbf{r} \times (-\nabla V).$$

(This is the rotational analog to Ehrenfest's theorem.)

(b) Show that $d\langle \mathbf{L} \rangle / dt = 0$ for any spherically symmetric potential. (This is one form of the quantum statement of **conservation of angular momentum**.)

Solution

Part (a)

Calculate the derivative of the expectation value of orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{L} \rangle &= \frac{d}{dt}\langle \Psi \,|\, \mathbf{L} \,|\, \Psi \rangle \\
&= \frac{d}{dt} \iiint\limits_{\text{all space}} \Psi^* \mathbf{L} \Psi \, d\mathcal{V} \\
&= \frac{d}{dt} \iiint\limits_{\text{all space}} \Psi^* \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j L_j \right) \Psi \, d\mathcal{V} \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{d}{dt} \iiint\limits_{\text{all space}} \Psi^* L_j \Psi \, d\mathcal{V} \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \frac{\partial}{\partial t}(\Psi^* L_j \Psi) \, d\mathcal{V} \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left(\frac{\partial \Psi^*}{\partial t} L_j \Psi + \Psi^* \frac{\partial L_j}{\partial t} \Psi + \Psi^* L_j \frac{\partial \Psi}{\partial t} \right) d\mathcal{V} \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{ \left(\frac{1}{i\hbar} \hat{H} \Psi \right)^* L_j \Psi + \Psi^* \left[\frac{\partial}{\partial t} (\mathbf{r} \times \mathbf{p})_j \right] \Psi + \Psi^* L_j \left(\frac{1}{i\hbar} \hat{H} \Psi \right) \right\} d\mathcal{V} \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{ \left(\frac{1}{-i\hbar} \Psi^* \hat{H}^\dagger \right) L_j \Psi + \Psi^* \left[\frac{\partial}{\partial t} \left(\sum_{k=1}^{3} \sum_{l=1}^{3} \varepsilon_{jkl} x_k p_l \right) \right] \Psi + \Psi^* L_j \left(\frac{1}{i\hbar} \hat{H} \Psi \right) \right\} d\mathcal{V}
\end{aligned}$$

Schrödinger's equation was used to eliminate the time derivatives of the wave function. After simplifying the integrand, substitute the Hamiltonian for a particle with mass m in a potential $V = V(x, y, z)$.

$$\frac{d}{dt}\langle \mathbf{L}\rangle = \sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left\{ \left(\frac{1}{-i\hbar}\Psi^* \hat{H}^\dagger\right) L_j \Psi + \Psi^* \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \cancelto{0}{\frac{\partial}{\partial t}(x_k p_l)}\right] \Psi + \Psi^* L_j \left(\frac{1}{i\hbar}\hat{H}\Psi\right)\right\} d\mathcal{V}$$

$$= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \left[\Psi^* \hat{H}(L_j\Psi) - \Psi^* L_j(\hat{H}\Psi)\right] d\mathcal{V}$$

$$= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left\{\hat{H}[(\mathbf{r}\times\mathbf{p})_j\Psi] - (\mathbf{r}\times\mathbf{p})_j(\hat{H}\Psi)\right\} d\mathcal{V}$$

$$= -\frac{1}{i\hbar}\sum_{j=1}^{3} \boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left\{\hat{H}\left[\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}x_k p_l\right)\Psi\right] - \sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}x_k p_l(\hat{H}\Psi)\right\} d\mathcal{V}$$

$$= -\frac{1}{i\hbar}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left[\hat{H}(x_k p_l\Psi) - x_k p_l(\hat{H}\Psi)\right] d\mathcal{V}$$

$$= -\frac{1}{i\hbar}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left\{\hat{H}\left[x_k\left(-i\hbar\frac{\partial}{\partial x_l}\right)\Psi\right] - x_k\left(-i\hbar\frac{\partial}{\partial x_l}\right)(\hat{H}\Psi)\right\} d\mathcal{V}$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left[\hat{H}\left(x_k\frac{\partial\Psi}{\partial x_l}\right) - x_k\frac{\partial}{\partial x_l}(\hat{H}\Psi)\right] d\mathcal{V}$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left[\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\left(x_k\frac{\partial\Psi}{\partial x_l}\right) - x_k\frac{\partial}{\partial x_l}\left(-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi\right)\right] d\mathcal{V}$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left[-\frac{\hbar^2}{2m}\nabla^2\left(x_k\frac{\partial\Psi}{\partial x_l}\right) + Vx_k\frac{\partial\Psi}{\partial x_l} + \frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi) - x_k\frac{\partial}{\partial x_l}(V\Psi)\right] d\mathcal{V}$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left[-\frac{\hbar^2}{2m}\nabla^2\left(x_k\frac{\partial\Psi}{\partial x_l}\right) + \cancel{Vx_k\frac{\partial\Psi}{\partial x_l}} + \frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi) - x_k\frac{\partial V}{\partial x_l}\Psi - \cancel{x_kV\frac{\partial\Psi}{\partial x_l}}\right] d\mathcal{V}$$

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left[-\frac{\hbar^2}{2m}\sum_{n=1}^{3}\frac{\partial^2}{\partial x_n^2}\left(x_k\frac{\partial\Psi}{\partial x_l}\right) + \frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi) - x_k\frac{\partial V}{\partial x_l}\Psi\right] d\mathcal{V}$$

Use the product rule to evaluate the derivatives.

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{L}\rangle &= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j \iiint_{\text{all space}} \Psi^*\left[-\frac{\hbar^2}{2m}\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(\frac{\partial x_k}{\partial x_n}\frac{\partial \Psi}{\partial x_l}+x_k\frac{\partial^2\Psi}{\partial x_n\partial x_l}\right)\right. \\
&\qquad\qquad \left.+\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)-x_k\frac{\partial V}{\partial x_l}\Psi\right]d\mathcal{V} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j \iiint_{\text{all space}} \Psi^*\left\{-\frac{\hbar^2}{2m}\left[\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(\frac{\partial x_k}{\partial x_n}\frac{\partial \Psi}{\partial x_l}\right)+\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(x_k\frac{\partial^2\Psi}{\partial x_n\partial x_l}\right)\right]\right. \\
&\qquad\qquad \left.+\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)-x_k\frac{\partial V}{\partial x_l}\Psi\right\}d\mathcal{V} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j \iiint_{\text{all space}} \Psi^*\left\{-\frac{\hbar^2}{2m}\left[\sum_{n=1}^{3}\frac{\partial}{\partial x_n}\left(\delta_{kn}\frac{\partial \Psi}{\partial x_l}\right)+\sum_{n=1}^{3}\left(\frac{\partial x_k}{\partial x_n}\frac{\partial^2\Psi}{\partial x_n\partial x_l}+x_k\frac{\partial^3\Psi}{\partial x_n^2\partial x_l}\right)\right]\right. \\
&\qquad\qquad \left.+\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)-x_k\frac{\partial V}{\partial x_l}\Psi\right\}d\mathcal{V} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j \iiint_{\text{all space}} \Psi^*\left\{-\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial x_k}\left(\frac{\partial \Psi}{\partial x_l}\right)+\sum_{n=1}^{3}\frac{\partial x_k}{\partial x_n}\frac{\partial^2\Psi}{\partial x_n\partial x_l}+\sum_{n=1}^{3}x_k\frac{\partial^3\Psi}{\partial x_n^2\partial x_l}\right]\right. \\
&\qquad\qquad \left.+\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)-x_k\frac{\partial V}{\partial x_l}\Psi\right\}d\mathcal{V} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j \iiint_{\text{all space}} \Psi^*\left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2\Psi}{\partial x_k\partial x_l}+\sum_{n=1}^{3}\delta_{kn}\frac{\partial^2\Psi}{\partial x_n\partial x_l}+x_k\frac{\partial}{\partial x_l}\left(\sum_{n=1}^{3}\frac{\partial^2\Psi}{\partial n^2}\right)\right]\right. \\
&\qquad\qquad \left.+\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)-x_k\frac{\partial V}{\partial x_l}\Psi\right\}d\mathcal{V} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j \iiint_{\text{all space}} \Psi^*\left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2\Psi}{\partial x_k\partial x_l}+\frac{\partial^2\Psi}{\partial x_k\partial x_l}+x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)\right]\right. \\
&\qquad\qquad \left.+\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)-x_k\frac{\partial V}{\partial x_l}\Psi\right\}d\mathcal{V} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\varepsilon_{jkl}\boldsymbol{\delta}_j \iiint_{\text{all space}} \Psi^*\left[-\frac{\hbar^2}{m}\frac{\partial^2\Psi}{\partial x_k\partial x_l}-\cancel{\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)}+\cancel{\frac{\hbar^2}{2m}x_k\frac{\partial}{\partial x_l}(\nabla^2\Psi)}-x_k\frac{\partial V}{\partial x_l}\Psi\right]d\mathcal{V}
\end{aligned}$$

Recall that the curl of a gradient is the zero vector: $\nabla \times \nabla f = \mathbf{0}$.

$$
\begin{aligned}
\frac{d}{dt}\langle \mathbf{L} \rangle &= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \iiint\limits_{\text{all space}} \Psi^* \left(-\frac{\hbar^2}{m}\frac{\partial^2 \Psi}{\partial x_k \partial x_l} - x_k \frac{\partial V}{\partial x_l}\Psi \right) d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \Psi^* \left[-\frac{\hbar^2}{m} \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j \frac{\partial}{\partial x_k}\left(\frac{\partial \Psi}{\partial x_l}\right) - \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl}\boldsymbol{\delta}_j x_k \frac{\partial V}{\partial x_l} \right) \Psi \right] d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \Psi^* \left[-\frac{\hbar^2}{m}(\nabla \times \nabla \Psi) - (\mathbf{r} \times \nabla V)\Psi \right] d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \Psi^* \left[-\frac{\hbar^2}{m}(\mathbf{0}) - (\mathbf{r} \times \nabla V)\Psi \right] d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \Psi^* [-(\mathbf{r} \times \nabla V)\Psi] \, d\mathcal{V} \\
&= \iiint\limits_{\text{all space}} \Psi^* [\mathbf{r} \times (-\nabla V)]\Psi \, d\mathcal{V} \\
&= \langle \Psi \,|\, \mathbf{r} \times (-\nabla V) \,|\, \Psi \rangle \\
&= \langle \mathbf{r} \times (-\nabla V) \rangle
\end{aligned}
$$

Therefore, since $\mathbf{N} = \mathbf{r} \times (-\nabla V)$,

$$\frac{d}{dt}\langle \mathbf{L} \rangle = \langle \mathbf{N} \rangle.$$

Part (b)

If the potential energy function is spherically symmetric, that is,

$$V = V(r),$$

then

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{L} \rangle = \langle \mathbf{N} \rangle &= \langle \mathbf{r} \times (-\nabla V) \rangle \\
&= \left\langle (r\hat{\mathbf{r}}) \times \left(-\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\overbrace{\frac{\partial V}{\partial \theta}}^{=0}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\overbrace{\frac{\partial V}{\partial \phi}}^{=0}\hat{\boldsymbol{\phi}} \right) \right\rangle \\
&= \left\langle -r\frac{dV}{dr}(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) \right\rangle \\
&= \left\langle -r\frac{dV}{dr}(\mathbf{0}) \right\rangle \\
&= \langle \mathbf{0} \rangle \\
&= \iiint_{\text{all space}} \Psi^*(\mathbf{0})\Psi \, d\mathcal{V} \\
&= \mathbf{0}.
\end{aligned}$$